Equational theories for scoped effects

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Introduction

Effects represent impure computation in a functional programming language, for example:

- nondeterminism,
- manipulating memory,
- taking input from a keyboard.

Moggi gave denotational semantics to effects using **strong monads** on the category of sets [Moggi'89,'91] $\Longrightarrow$ Monads for programming in Haskell.

Introduction: Plotkin's and Power's framework of algebraic effects

Idea: effects are realized by operations and equations that **generate** a monad, but are not identified with the monad.

- Operations produce the effects (e.g. reading and writing to memory, nondeterministic choice).
- Equations specify the behaviour of these operations.

Theorem [Lawvere, Linton]

To give a finitary monad on the category of sets is equivalent to giving an algebraic theory (in the sense of universal algebra, or Lawvere theories).

Introduction: Plotkin's and Power's framework of algebraic effects

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- Equations specify the behaviour of these operations.

Example: the List monad is presented by two operations:

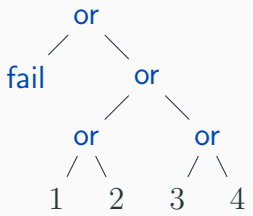
- $+\!\!+$, list concatenation, which is associative, and
- $[]$ the empty list, unit element for $+\!\!+$.

Introduction: algebraic vs scoped operations

Algebraic operations commute with sequencing (monadic bind).

Example: nondeterministic choice and failure.
The following two terms perform the same computation:

$$\mathsf{or}\big(\mathsf{fail},\ \mathsf{or}(1 \gg\!= \lambda x.\, \mathsf{or}(x, x+1),\ 3 \gg\!= \lambda x.\, \mathsf{or}(x, x+1))\big) \qquad \mathsf{or}\big(\mathsf{fail},\ \mathsf{or}(1,3)\big) \gg\!= \lambda x.\, \mathsf{or}(x, x+1)$$

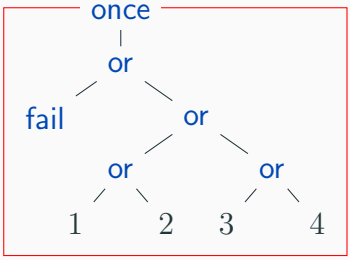


Introduction: algebraic vs scoped operations

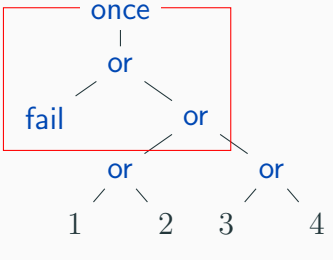
Scoped operations do not commute with sequencing [Wu et al'14].

Example: once chooses the first branch that does not fail.
The scope of once is delimited.

$\mathsf{once}(\mathsf{or}(\mathsf{fail},\ \mathsf{or}(\mathsf{or}(1,2),\mathsf{or}(3,4)))) \rightsquigarrow 1$



$\mathsf{once}(\mathsf{or}(\mathsf{fail},\ \mathsf{or}(1,3))) \gg\!= \lambda x.\ \mathsf{or}(x, x+1) \rightsquigarrow \mathsf{or}(1,2)$



Introduction: algebraic vs scoped operations

Scoped operations do not commute with sequencing [Wu et al'14].

Example: catching exceptions

$$\mathsf{catch}(x, y) \gg\!= k \neq \mathsf{catch}(x \gg\!= k, y \gg\!= k)$$

Scoped effects don't fit in the framework of algebraic effects.

In this talk

[Lindley et al ESOP'24, Matache et al TOPLAS'25]

- Clarify what scoped effects are using monads and algebraic theories.
- Recover complete equational reasoning for scoped effects.

Outline

Algebraic theories

An **algebraic theory** $\mathcal{T} = (\Sigma, E)$ consists of a signature $\Sigma : \mathcal{O} \to \mathbb{N}$ of operations, and a set of equations E.

Example: explicit nondeterminism (backtracking) [Plotkin & Pretnar'09, '13]

$\mathsf{or} : 2$ choice

$\mathsf{fail} : 0$ failure

$$x, y, z \vdash \mathsf{or}(x, \mathsf{or}(y, z)) = \mathsf{or}(\mathsf{or}(x, y), z)$$

$$x \vdash \mathsf{or}(\mathsf{fail}, x) = \mathsf{or}(x, \mathsf{fail}) = x$$

In direct style:

$$\underline{\mathsf{or}} : \mathsf{unit} \to \mathsf{bool} \qquad \underline{\mathsf{fail}} : \mathsf{unit} \to \mathsf{empty}$$

Translation between direct style and algebraic operations:

$$\mathsf{or}(x, y) = \mathsf{if}\ \underline{\mathsf{or}}()\ \mathsf{then}\ x\ \mathsf{else}\ y \qquad \underline{\mathsf{or}}() = \mathsf{or}(\mathsf{true}, \mathsf{false})$$

Algebraic theories

Example: explicit nondeterminism (backtracking) [Plotkin & Pretnar'09, '13]

$\mathsf{or} : 2$	choice	$x, y, z \vdash \mathsf{or}(x, \mathsf{or}(y, z)) = \mathsf{or}(\mathsf{or}(x, y), z)$
$\mathsf{fail} : 0$	failure	$x \vdash \mathsf{or}(\mathsf{fail}, x) = \mathsf{or}(x, \mathsf{fail}) = x$

Model of an algebraic theory = a set (carrier) + interpretations for the operations, satisfying the equations.

Fix a set A. The intended model for the theory of nondeterminism is:

- Carrier: the set $\mathsf{List}(A)$
- Operations:

$$[\![\mathsf{or}]\!] : \mathsf{List}(A)^2 \to \mathsf{List}(A), \qquad [\![\mathsf{or}]\!](l_1, l_2) = l_1 +\!\!+ l_2$$
$$[\![\mathsf{fail}]\!] : 1 \to \mathsf{List}(A), \qquad [\![\mathsf{fail}]\!]() = []$$

Algebraic theories

- $\mathsf{List}(A)$ is the **free model** on A of the theory of nondeterminism.
- $\mathsf{List}(-)$ extends to a monad on Set (part of the finitary monad - algebraic theory correspondence).
- Consequence of freeness: the equations in the theory are **sound and complete** with respect to the $\mathsf{List}(A)$ model.

Algebraic theories

- $\mathsf{List}(A)$ is the **free model** on A of the theory of nondeterminism.
- $\mathsf{List}(-)$ extends to a monad on Set (part of the finitary monad - algebraic theory correspondence).
- The List monad **supports** the operations from the theory, meaning e.g.:

$$[\![\mathsf{or}]\!] : \mathsf{List}(A) \times \mathsf{List}(A) \to \mathsf{List}(A)$$

is natural in A and commutes with $\gg\!=$ of the List monad, e.g.

$$\big([\![\mathsf{or}]\!]([1],[3]) \gg\!= k\big) = [\![\mathsf{or}]\!]([1] \gg\!= k,\ [3] \gg\!= k) = [1,2,3,4]$$

where $k = \lambda x.[x, x+1]$.

Outline

What is a scoped effect? Four different views

(1) A **scoped operation** is a handler for an algebraic effect. Semantically, an algebra for a monad on Set. [Plotkin & Pretnar'09, '13]

(2) Given a monad T on Set, a **scoped operation** sc : k is a natural transformation $T^k \to T$. [Wu et al'14, Yang & Wu ICFP'23].

(3) A **scoped effect** is determined by an endofunctor on $\mathsf{Set}^{|\mathbb{N}|}$, ($|\mathbb{N}|$ is a discrete category), and taking the free monad on this endofunctor. [Piróg et al LICS'18]

(4) A **scoped effect** is determined by an endofunctor on $\mathsf{Set}^{\mathsf{FinSet}}$, and taking the free monad on this endofunctor. [Wu et al'14, Yang et al ESOP'22]

Nondeterminism with once as a scoped effect [Wu et. al.'14]

(2) Given a monad T on Set, a **scoped operation** sc : k is a natural transformation $T^k \to T$. [Wu et al'14, Yang & Wu ICFP'23].

Recall that once chooses the first non-failing branch of a computation.
It fits the definition of a scoped operation.

The List monad supports the natural transformation:

$$[\![\text{once}]\!] : \mathsf{List}(A) \to \mathsf{List}(A) \qquad [\![\text{once}]\!]([a, \dots]) = [a] \qquad [\![\text{once}]\!]([]) = []$$

which does not commute with $\gg\!=$

$$\big([\![\text{once}]\!]([\![\text{or}]\!]([1],[3]))\big) \gg\!= k = \big([1] \gg\!= k\big) = [1,2]$$

$$[\![\text{once}]\!]\big([\![\text{or}]\!]([1],[3]) \gg\!= k\big) = [\![\text{once}]\!]([1,2,3,4]) = [1]$$

where $k = \lambda x.\, [x, x+1]$.

Our proposal: a scoped effect is a theory

(5) A **scoped effect** is an algebraic theory of a suitable kind.

This definition provides **equational reasoning** for scoped effects.

- This definition agrees with (2) and (3) for examples of scoped effects:
  - nondeterminism (backtracking) with once,
  - exception catching,
  - state with local values
  - nondeterminism (backtracking) with cut.
- Theorems connecting definition (5) with (2) and (3) in general.

Outline

Scoped effects as parameterized algebraic theories

Parameters = names of scopes

A scoped effect = a theory with **ordered, linear** parameters

Example: nondeterminism with once

$\mathsf{or}(x, y)$	choice
fail	failure
$\mathsf{once}(a.x(a))$	open a new scope named a, continue as $x(a)$; a is bound
$\mathsf{close}(a;\ y)$	close the scope a, y cannot use a anymore; a is free

A term has two contexts. The context of parameters is ordered and linear.

Example:

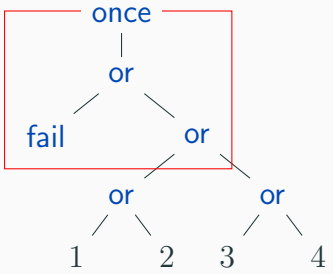
$$x : 1 \mid b \vdash \mathsf{close}(b;\ \mathsf{once}(a.x(a)))$$

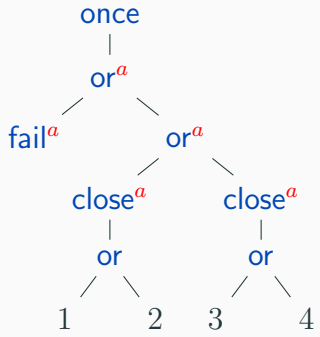
Nondeterminism with once as a parameterized theory

Closing a scope becomes an explicit operation.

$$\mathsf{once}(\mathsf{or}(\mathsf{fail},\ \mathsf{or}(1,3))) \gg\!= \lambda x.\, \mathsf{or}(x, x+1) \rightsquigarrow \mathsf{or}(1,2)$$

$$\mathsf{once}\big(a.\mathsf{or}(\mathsf{fail}, \mathsf{or}(\mathsf{close}(a;\ \mathsf{or}(1,2)), \mathsf{close}(a;\ \mathsf{or}(3,4)))\big)$$





Equations for nondeterminism with once

Explicit nondeterminism

$$\mathsf{or}(x, \mathsf{or}(y, z)) = \mathsf{or}(\mathsf{or}(x, y), z)$$

$$\mathsf{or}(\mathsf{fail}, x) = \mathsf{or}(x, \mathsf{fail}) = x$$

Once/close

$$\mathsf{once}(a.\mathsf{fail}) = \mathsf{fail} \quad (1)$$

$$\mathsf{once}(a.\mathsf{or}(x(a),\ x(a))) = \mathsf{once}(a.x(a)) \quad (2)$$

$$\mathsf{once}\big(a.\mathsf{or}(\mathsf{close}(a;\ x),\ y(a))\big) = x \quad (3)$$

We can derive:

$$\mathsf{once}(a.\mathsf{close}(a;\ x)) = x$$

We can prove using the equations:

$$\mathsf{once}\big(a.\mathsf{or}(\mathsf{fail},\ \mathsf{or}(\mathsf{close}(a;\ \mathsf{or}(1, 2)),\ \mathsf{close}(a;\ \mathsf{or}(3, 4))))\big) = \mathsf{or}(1, 2)$$

Parameterized algebraic theories [Staton FoSSaCS'13]

- Extend plain algebraic theories with binding.
- Have a correspondence with certain strong monads on a functor category. In our case: $\mathsf{Set}^{|\mathbb{N}|}$, where $|\mathbb{N}|$ is discrete.
  Via the free model construction.
- Provide an equational reasoning system.

Our proposed definition

A **scoped theory** is a parameterized theory with linear, ordered parameters with certain restrictions on the signature (to match examples).

E.g. operations can bind at most one parameter in each argument;
there is a close operation that consumes one parameter.

Models for explicit nondeterminism with once

Model = an object from $\mathsf{Set}^{|\mathbb{N}|}$ (carrier) + interpretations for the operations.

The **free model** on $A = (A_0, \emptyset, \emptyset, \dots) \in \mathsf{Set}^{|\mathbb{N}|}$ has:

- Carrier: the sequence $TA(n) = \mathsf{List}^{n+1}(A_0)$, for $n \in \mathbb{N}$
- Operations:

$$
\begin{array}{lll}
[\![\mathsf{once}]\!]_n : TA(n+1) \to TA(n) & [\![\mathsf{once}]\!]_n([a, \dots]) = a, & [\![\mathsf{once}]\!]_n([]) = [] \\
[\![\mathsf{close}]\!]_n : TA(n) \to TA(n+1) & [\![\mathsf{close}]\!]_n(l) = [l] & \\
[\![\mathsf{or}]\!]_n : TA(n)^2 \to TA(n) & [\![\mathsf{or}]\!]_n(l_1, l_2) = l_1 +\!\!+\, l_2 & \\
[\![\mathsf{fail}]\!]_n : 1 \to TA(n) & [\![\mathsf{fail}]\!]_n() = [] &
\end{array}
$$

Recall: $\mathsf{once}(a.x(a))$ $\mathsf{close}(a;\, y)$ $\mathsf{or}(x, y)$ fail

Relation to previous definitions of scoped effects

(2) Given a monad T on Set, a **scoped operation** sc : k is a natural transformation $T^k \to T$. [Wu et al'14, Yang & Wu ICFP'23].

Theorem

From the monad on $\mathsf{Set}^{|\mathbb{N}|}$ generated by the theory of nondeterminism with once, we can recover the List monad on Set and the scoped operation $[\![\mathsf{once}]\!] : \mathsf{List} \to \mathsf{List}$.

At least two inequivalent scoped theories that recover $(\mathsf{List}, [\![\mathsf{once}]\!])$. We chose an intuitive one.

More generally, from Def (2) we can always construct a (maximal) parameterized theory, then recover the data of Def (2) from it.

Relation to previous definitions of scoped effects

(3) A **scoped effect** is determined by an endofunctor on $\mathsf{Set}^{|\mathbb{N}|}$, ($|\mathbb{N}|$ is a discrete category), and taking the free monad on this endofunctor. [Piróg et al LICS'18]

A model is an algebra for this free monad.

Theorem

The **free model** on $A = (A_0, \emptyset, \emptyset, \dots) \in \mathsf{Set}^{|\mathbb{N}|}$ for the theory of nondeterminism with once, is the monad algebra from [Piróg et al LICS'18].

We gave a framework in which **complete equational axiomatization** of their model is possible.

More generally, given a scoped theory with no equations, the induced monad on $\mathsf{Set}^{|\mathbb{N}|}$ is isomorphic to the free monad of [Piróg et al LICS'18].

Outline

Parameterized algebraic theories [Staton FoSSaCS'13, LICS'13, POPL'15]

Different applications of parameterized theories, by varying the structural rules of parameters:

Example	Parameters	Models in
name generation	names	$\mathsf{Set}^{\mathsf{Fin}}$
local state	location names	
π-calculus (fragment)	communication channels	
predicate logic	individuals	
quantum computation	qubits (linear)	$\mathsf{Set}^{\mathsf{Bij}}$
scoped effects [ESOP'24, TOPLAS'25]	scopes (ordered, linear)	$\mathsf{Set}^{\lvert\mathbb{N}\rvert}$
POSIX-like fork [POPL'26]	thread IDs	$\mathsf{Set}^{\mathsf{FinRel}}$

A parameterized theory of threads — Next Saturday, ACV workshop

Parameters = thread IDs. (Not used linearly anymore.)

Operations:

fork$(a.x(a),\ y)$	x is the parent thread; might use a y is the child thread; cannot use a a is the ID of child; a is bound
wait$(a;\ x)$	wait for the thread named a to finish, continue as x
stop	end the current thread, unblock all threads waiting for it
act$_\sigma(x)$	perform observable action σ, continue as x

We axiomatize these operations with 9 equations.

A parameterized theory of threads — Example equations

The term $\mathsf{wait}(a;\, \mathsf{stop})$ acts as a unit for fork:

$$\mathsf{fork}(a.\mathsf{wait}(a; \mathsf{stop}), x) = x \qquad \mathsf{fork}(b.x(b), \mathsf{wait}(a; \mathsf{stop})) = x(a)$$

Operations wait and fork commute:

$$\mathsf{wait}(b;\, \mathsf{fork}(a.x(a),\, y)) = \mathsf{fork}(a.\mathsf{wait}(b;\, x(a)),\, \mathsf{wait}(b;\, y))$$

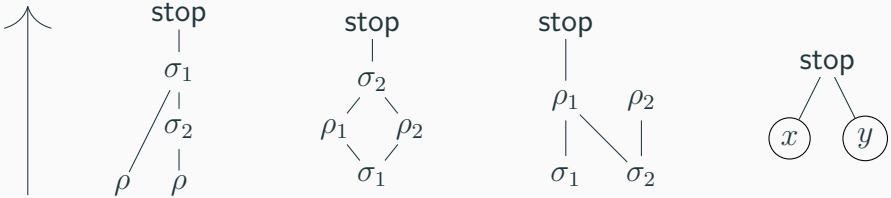
Performing an action can be delegated to a child thread:

$$\mathsf{act}_\sigma(x) = \mathsf{fork}(a.\mathsf{wait}(a, x),\, \mathsf{act}_\sigma(\mathsf{stop}))$$

A parameterized theory of threads

The elements of the monad obtained from the parameterized theory are **partial orders with labels** (pomsets).

Concurrent programs denote pomsets, for example:



We use the monad to give an adequate, fully abstract at first-order, semantics to a concurrent lambda calculus.

Summary

Scoped effects are operations that do not commute with monadic bind.

Our proposed definition

A **scoped effect** is a certain kind of parameterized algebraic theory.

We illustrate this definition with four examples:

- nondeterminism (backtracking) with once,
- exception catching,
- state with local values
- nondeterminism (backtracking) with cut.

Our definition provides **complete equational reasoning** for the examples.

Summary

Scoped effects are operations that do not commute with monadic bind.

Our proposed definition

A **scoped effect** is a certain kind of parameterized algebraic theory.

Theorems connecting to two other proposed definitions of scoped effects.

Future work:

- Axiomatizing more examples of scoped effects.
- Programming with scoped effects in direct style.
- Handlers for parameterized effects: ongoing work on an Idris library [Sigal et al HOPE'24]